§3.1—Understanding the Limit

What do you see below?

We are building the “House of Calculus,” one side at a time . . . and we need a solid FOUNDATION.
Going where we can’t go

Example 1:
If \( f(x) = x^2 \), go to \( x = 2 \) and see what \( y \)-value you end up at. Graph \( f(x) = x^2 \) and the coordinate of your location. Is there any \( x \)-value on this function where you can’t go?

Note: when I refer to “location,” I am talking about the \( x \)-value. When I talk about “point” or “coordinate,” I’m referring to the ordered pair \((x, f(x))\)

Example 2:
If \( g(x) = \frac{2x^2 - 4x + 2}{x - 1} \), go to \( x = 1 \) and see what the \( y \)-value is there. If you cannot get there, what’s going on there? Are you curious? How can we explore what is going on there if we aren’t able to get there?

This interruption to the flow of the graph is called a **removable point discontinuity**, or hole.
It is now easy to convince you that simply being able to evaluate a function at different values is insufficient for understanding the behavior of some types of functions, namely those that have domain restrictions. There is, therefore, a need to come up with another method that will circumvent the possibility of going directly to a location, but rather approaching that location from either side of it. This is the limit, and it has its own notation as you will see.

**The Limit is a Notion of Motion**

How close would you like to get to the edge of the bridge in order to convince yourself of what is going on there?

What if you were to inch to the edge of the bridge on the other side?

What if the bridge went all the way across?

The existence or non-existence of the bridge is irrelevant to the existence of the gorge. This is exactly why the limit is so important. It gives us a way to talk about the activity “at” a point whether or not the graph exists there or not.

**Example 3:**
Let’s say a road is defined by the graph of \( f(x) = 2x + 2 \). There is a gorge at \( x = 1 \). We want to know what the \( y \)-values are approaching as the \( x \)-values are approaching \( x = 1 \) from both sides of the \( x \)-coordinate of the gorge. Use your calculator to explore what is happening as you approach \( x = 1 \) from both sides of \( x = 1 \). Write your results as one-sided limits.

If and only if a graph approaches the same \( y \)-value from both sides, we say the limit, in general, exists. In this case, we’d write the results found above as:
Here’s a summary:

**Theorem:**

\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \iff \lim_{x \to c} f(x) = L
\]

**Definition:**

A function \( f(x) \) is **continuous at a point** \( x = c \) if

1. \( f(c) \) is defined
2. \( \lim_{x \to c} f(x) \) exists
3. \( f(c) = \lim_{x \to c} f(x) \)

Here’s an alternative version of the definition above that is more in line with the “road-bridge-road” analogy.

**Definition (alternate):**

A function \( f(x) \) is **continuous at a point** \( x = c \) if

\[
\lim_{x \to c^{-}} f(x) = f(c) = \lim_{x \to c^{+}} f(x)
\]

that is

road = bridge = road

This definition is great for determining whether a function is continuous (which we’ll do here shortly), but if we state the result a little differently, we get a way to evaluate limits.

**If a function is continuous at a point, the function value and the limit value are the same at that point!**

**Example 4:**

Discuss the continuity of \( f(x) = x^2 - 5 \), then use that fact to find \( \lim_{x \to 2} (x^2 - 5) \). Draw a picture of your results.
Example 5:
Let \( f(x) = \frac{2x^2 - 2}{x - 1} \). Find \( \lim_{{x \to 1}} f(x) \) using your calculator’s analytic capabilities.

Those of you who still remember from Precalculus how to sketch rational functions with holes might realize there’s an algebraic method for finding the limit without relying on the calculator.

Example 6:
Let \( f(x) = \frac{2x^2 - 2}{x - 1} \). Find \( \lim_{{x \to 1}} f(x) \) using an algebraic method.

It’s worth noting that if we were interested in evaluating the limit of \( f(x) \) other than at \( x = 1 \), since \( f(x) \) is continuous everywhere else, we could use direct substitution. That is, for \( x \neq 1 \), \( \lim_{{x \to c}} f(x) = f(c) = g(c) \).

Example 7:
Evaluate \( \lim_{{x \to -2}} \frac{x^3 + 3x^2 - 4x - 12}{x^2 + x - 2} \) algebraically. Verify graphically and numerically on your calculator.
Example 8:
Given the function

\[ f(x) = \begin{cases} 
  x + 2, & x < -2 \\
  \frac{x^2}{2}, & -2 \leq x \leq 2 \\
  2, & x > 2
\end{cases} \]

(a) Sketch the graph of the function

(b) Using the 3-step definition of continuity, discuss the continuity of \( f(x) \) at \( x = 2 \).

(c) Using the 3-step definition of continuity, discuss the continuity of \( f(x) \) at \( x = -2 \).

When the two one-sided limits exist but are different values, we get what’s called a non-removable jump discontinuity.
We’ve discussed two ways in which a graph can be interrupted and, thus, two types of discontinuities: removable points and non-removable jumps. There is a third type of discontinuity with which everyone from Precalculus is very familiar: Vertical Asymptotes. Officially called non-removable infinite discontinuities, the limit will always fail to exist these locations, but for a different reason than why the limit failed to exist at a jump. As we approach a vertical asymptote from either side, there are only two options, go up forever to infinity or go down forever to negative infinity.

Definition:

\[
\lim_{x \to c^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = -\infty
\]

if and only if

there exists a **vertical asymptote** at \( x = c \).

**Example 9:**
Remembering you parent functions from Precal, evaluate the following. P.S. Don’t use a calculator if you don’t have to. P.S.S. You don’t have to.

I. \( h(x) = \ln x \)
   (a) \( h(0) = \)
   (b) \( \lim_{x \to 0^-} h(x) = \)
   (c) \( \lim_{x \to 0^+} h(x) = \)
   (d) \( \lim_{x \to 0} h(x) = \)

II. \( f(x) = \frac{1}{x-3} \)
    (a) \( f(3) = \)
    (b) \( \lim_{x \to 3^-} f(x) = \)
    (c) \( \lim_{x \to 3^+} f(x) = \)
    (d) \( \lim_{x \to 3} f(x) = \)

III. \( g(x) = \frac{1}{(x+2)^2} \)
     (a) \( g(-2) = \)
     (b) \( \lim_{x \to -2^-} g(x) = \)
     (c) \( \lim_{x \to -2^+} g(x) = \)
     (d) \( \lim_{x \to -2} g(x) = \)

There is one last type of discontinuity that can be a little triggy. It’s called an **oscillating discontinuity**.

**Example 10:**
Examine \( f(x) = \sin \left( \frac{1}{x} \right) \) in the vicinity of \( x = 0 \). Verify your results by graphing the function and zooming in around \( x = 0 \). For kicks and giggles, analyze the end behavior of this same function. Paradoxical results?!
Hopefully you’re getting comfortable with this new idea of a limit value and how it is categorically different from a function value. Additionally, you’ve no doubt become aware of how important both the limit value and the function value are important to the idea of continuity at a point.

When looking for a limit value at \( x = c \), imagine that you’ve got a thick vertical line covering up \( x = c \) with only the graph showing on either side of \( x = c \). You are now looking to see what \( y \)-value(s) the graph is approaching on either side of \( x = c \). If the graphs appear to be approaching the same \( y \)-value, the limit exists and is that \( y \)-value. Otherwise, the limit does not exist there.

When looking for a function value at \( x = c \), imagine that you’ve got shudders up on either side of \( x = c \) with only the vertical sliver at \( x = c \) visible between them. You are now looking for the dot or the piece of the graph that exists in that narrow sliver. If it exists, the \( y \)-value of the dot is the function value \( f(c) \).

Here’s what the graph above looks like without the visual aids above.

Both the limit and function values exist at \( x = c \), but because they are different, the graph is not continuous at \( x = c \) and has a removable point discontinuity at \( x = c \).
Let’s look at messy graph of $f(x)$ with many discontinuities. Call it “graphus interruptus.”

**Example 11:**
Discuss the limits, function values, and continuity at various $x$-values in the graph above.